# Numerical analysis of an energy-like minimization method to solve a parabolic Cauchy problem with noisy data

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**Abstract**

This paper is concerned with solving Cauchy problem for parabolic equation by minimizing an energy-like error functional and by taking into account noisy Cauchy data. After giving some fundamental results, numerical convergence analysis of the energy-like minimization method is carried out and leads to an adapted stopping criterion depending on noise rate for the minimization process. Numerical experiments are performed and confirm theoretical convergence order and the good behavior of the minimization process.

## 1 Introduction

The Cauchy problem considered here consists of solving a parabolic partial differential equation on a domain for which over-specified boundary conditions are given on a part of its boundary. It entails solving a data completion problem and identifying the missing boundary conditions on the remaining part of the boundary. This kind of problem is encountered in many industrial, engineering and biomedical applications.

Since J.Hadamard's works [1], the Cauchy problem is known to be ill-posed and considerable numerical instability may occur during the resolution process. It provides researchers with an interesting challenge for carrying out numerical procedures to approximate the solution of the Cauchy problem in the specific case of noisy data. Many theoretical and applied works have been dedicated to this subject, using iterative methods [2], regularization methods [3, 4], quasi-reversibility methods [5] and minimal error methods [6, 7, 8].

In this paper, we focus on a method introduced in [9, 10, 11, 12, 13] based on the minimization of an energy-like functional. More precisely, we introduce two distinct fields, each of which fulfills one of the over specified boundary conditions. They are therefore solutions of two well-posed problems. Next, an energy-like error functional is introduced to measure the gap between these two fields. If the Cauchy problem solution exists and is unique, it is obtained when the functional reaches its minimum. Then, the resolution of the ill-posed Cauchy problem is achieved by successive resolutions of well-posed problems. This method provides promising results. Nevertheless, like many other methods, it becomes unstable in the case of noisy data. To overcome this numerical instability, we propose an adequate stopping criterion parametrized by the noise rate deduced by numerical convergence analysis. This analysis has already been performed for elliptic Cauchy problems in [14].

The outline of the paper is as follows. In section 2, we give the Cauchy problem and report classical theoretical results. In section 3, we formulate the Cauchy problem as a data completion problem and introduce the related minimization problem. In sections 4 and 5, we present finite element and time discretization, convergence analysis and the study of noise effects for the minimization problem. An a priori error estimate is then given, taking into account data noise, and a stopping criterion is proposed to control the instability of the minimization process. Finally, the numerical procedure and results are presented.

## 2 Statement of problem

We consider a Lipschitz bounded domain $\Omega$ in $\mathbb{R}^d$, $d = 2, 3$ with $n$ being the outward unit normal to the boundary $\Gamma = \partial\Omega$. Let us assume that $\Gamma$ is partitioned into two parts, $\Gamma_u$ (for *u*nknowns) and $\Gamma_m$ (for *m*easurements), of the non-vanishing measurement, such that $\Gamma_u \cap \Gamma_m = \emptyset$.

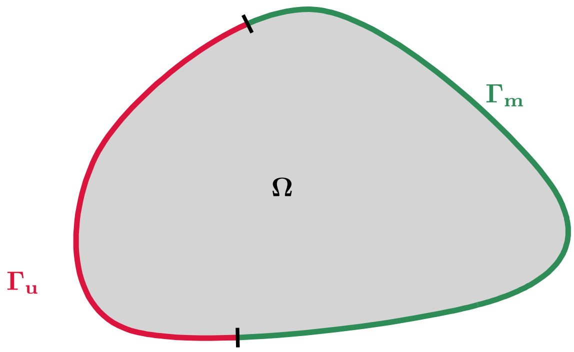



Figure 1: An example of geometry

The most common problem consists in solving the heat transfer equation in a given domain $\Omega$ and a time interval $[0, D]$, assuming temperature distribution and heat flux are given over the accessible region of the boundary. We denote for $D > 0$

$$Q = \Omega \times ]0, D[, \quad \Sigma_u = \Gamma_u \times ]0, D[, \quad \Sigma_m = \Gamma_m \times ]0, D[.$$

Given an initial temperature $u_0$ in $\Omega$, a source term $\tilde{f}$, a conductivity field $\tilde{k}$, a density $\rho$ and a heat capacity $c$ in $Q$, a flux $\tilde{\phi}$ and the corresponding temperature $T$ on $\Sigma_m$, the aim is to identify the corresponding flux and temperature on $\Sigma_u$. The nondimensionalized Cauchy problem is then written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot \big(k(x)\nabla u\big) = f \text{ in } Q \\ k(x)\nabla u \cdot n = \phi \text{ on } \Sigma_m \\ u = T \text{ on } \Sigma_m \\ u(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases} \tag{1}$$

where $k(x) = \tilde{k}(x)/\rho c$, $f = \tilde{f}/\rho c$ and $\phi = \tilde{\phi}/\rho c$.

A problem is well-posed according to Hadamard (see [1, 15, 3]) if it fulfills the following properties : the uniqueness, existence and stability of the solution. The extended Holmgren theorem relating to Sobolev spaces (see [15]) guarantees uniqueness under regularity assumptions for the solution of the Cauchy problem. Since the well known Cauchy-Kowalevsky theorem (see [16]) is applicable only in the case of analytical data, the existence of this solution threefore depends on the verification of a compatibility condition difficult to formulate explicitly. In addition to the fact that for one fixed datum, the set of compatible data is dense within the full set of data (see [17]), this compatibility condition implies that the stability assumption is not satisfied in the sense that the dependence of solution $u$ of (1) on data $(\phi, T)$ is not continuous. Hereafter, we assume that data $(\phi, T)$ in (1) are compatible.

*A few notations* : Let $x$ be a generic point of $\Omega$. The space of squared integrable functions $L^2(\Omega)$ is endowed with a natural inner product written as $(\cdot, \cdot)_{0,\Omega}$. The associated norm is written as $\|\cdot\|_{0,\Omega}$. We note $H^p(\Omega)$ the Sobolev space of functions of $L^2(\Omega)$ for which their $p$-th order and lower derivatives are also in $L^2(\Omega)$. Its norm and semi norm are written as $\|\cdot\|_{p,\Omega}$ and $|\cdot|_{p,\Omega}$ respectively. Moreover, let $u=(u_1,u_2)\in (H^p(\Omega))^2$, the semi-norm of this space is written $|||u|||_{p,\Omega} = \left(|u_1|^2_{p,\Omega}+|u_2|^2_{p,\Omega}\right)^{1/2}$. Let $\gamma\subset\Gamma$, we define the space $H^1_{0,\gamma}(\Omega)=\{v\in H^1(\Omega); v_{|\gamma}=0\}$ and $H^{1/2}_{00}(\gamma)$ is the space of restrictions to $\gamma$ of the functions of $H^{1/2}(\Omega)=tr\big(H^1(\Omega)\big)$. Its topological dual is written as $H^{-1/2}_{00}(\gamma)=\big(H^{1/2}_{00}(\gamma)\big)'$. The associated norms are written as $\|\cdot\|_{1/2,00,\gamma}$ and $\|\cdot\|_{-1/2,00,\gamma}$ respectively and $\langle\cdot,\cdot\rangle_{1/2,00,\gamma}$ states for the duality inner product. Now, let $t$ be the time variable. We denote by $L^2(0,D;F)$ the space of squared integrable functions in $[0,D]$ with values in $F$, where $F$ is a normed functional space. In the same way, $\mathscr{C}^n(0,D;F)$ defines the space of $n$ times continuously derivable functions in $[0,D]$ with values in $F$. The space of distributions in $]0,D[$ is written as $\mathscr{D}'(]0,D[)$. In the sequel, $C$ indicates a positive generic constant.

# 3 Energy-like minimization method

Let $f\in L^2\left(0,D;L^2(\Omega)\right)$, $k(x)\in L^\infty(\Omega)$ positive, $\phi\in L^2\big(0,D;H^{-1/2}_{00}(\Gamma_m)\big)$ and $T\in L^2\big(0,D;H^{1/2}_{00}(\Gamma_m)\big)$. The Cauchy problem can be written as a data completion problem:

Find $(\varphi,\xi)\in L^2\left(0,D;H^{-1/2}_{00}(\Gamma_u)\times H^{1/2}_{00}(\Gamma_u)\right)$ such that $u\in L^2\left(0,D;H^1(\Omega)\right)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla\cdot\big(k(x)\nabla u\big) = f \text{ in } Q \\ u = T,\ k(x)\nabla u\cdot n = \phi \text{ on } \Sigma_m \\ u=\xi,\ k(x)\nabla u\cdot n = \varphi \text{ on } \Sigma_u \\ u(\cdot,0)=u_0 \text{ in } \Omega. \end{cases} \tag{2}$$

**Remark 1** : We note that in the case $\bar\Gamma_u\cap\bar\Gamma_m=\emptyset$, as given in figure 2 illustrating the ring numerical tests of section 6.2, spaces $H^{-1/2}(\Gamma_u)\times H^{1/2}(\Gamma_u)$ and $H^{-1/2}(\Gamma_m)\times H^{1/2}(\Gamma_m)$ for the unknowns and the data respectively, would be more appropriate. Nevertheless, the general functional framework is not restrictive because spaces $H^s_{00}(\Gamma_u)$ and $H^s_{00}(\Gamma_m)$ are dense in $H^s(\Gamma_u)$ and $H^s(\Gamma_m)$, respectively, for $s=\pm 1/2$.

Following [12], we now introduce two distinct fields $u_1$ and $u_2$ which are the solutions of well posed problems differentiated by their boundary conditions. We attribute to each of them one datum on $\Sigma_m$ and one unknown on $\Sigma_u$. Then, we obtain

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \nabla\cdot\big(k(x)\nabla u_1\big) = f \text{ in } Q \\ u_1 = T \text{ on } \Sigma_m \\ k(x)\nabla u_1\cdot n = \eta \text{ on } \Sigma_u \\ u_1(\cdot,0)=u_0 \text{ in } \Omega, \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{\partial u_2}{\partial t} - \nabla\cdot\big(k(x)\nabla u_2\big) = f \text{ in } Q \\ u_2 = \tau \text{ on } \Sigma_u \\ k(x)\nabla u_2\cdot n = \phi \text{ on } \Sigma_m \\ u_2(\cdot,0)=u_0 \text{ in } \Omega. \end{cases} \tag{4}$$

We denote $a_i(\cdot,\cdot)$ and $l_i(\cdot)$, $i=1,2$ the bilinear and linear forms associated to the weak forms of the problems (3) and (4) respectively. They are given by

$$a_i(\tilde u_i(t),v) = \int_\Omega k(x)\nabla\tilde u_i(t)\nabla v\ dx, \quad \text{for } i=1,2, \tag{5}$$

$$l_1(v;t) = \int_\Omega f(t)v\ dx - \frac{d}{dt}(\bar u_1(t),v_1)_H - a_1(\bar u_1(t),v) + \langle\eta(t),v\rangle_{1/2,00,\Gamma_u}, \tag{6}$$

$$l_2(v;t) = \int_\Omega f(t)v\ dx - \frac{d}{dt}(\bar u_2(t),v_1)_H - a_2(\bar u_2(t),v) + \langle\phi(t),v\rangle_{1/2,00,\Gamma_m}, \tag{7}$$

where $\bar u_1(t)$ and $\bar u_2(t)$ are the lifting of the extended Dirichlet conditions $T(t)$ and $\tau(t)$ respectively and $\tilde{u}_i = u_i - \bar{u}_i,\ i = 1, 2$. We have by summation the following weak problem:

$$\begin{array}{rl} \text{Find} & u = (\tilde{u}_1, \tilde{u}_2) \in L^2(0, D; V) \cap \mathscr{C}^0(0, D; H) \text{ such that} \\ & \dfrac{d}{dt}(u(t), v)_H + a(u(t), v) = L(v; t),\ \forall\ v = (v_1, v_2) \in V \text{ in } \mathscr{D}'(]0, D[) \\ & u(., 0) = u_{00} = (\tilde{u}_{10}, \tilde{u}_{20}), \\ \text{with} & a(u(t), v) = a_1(\tilde{u}_1(t), v_1) + a_2(\tilde{u}_2(t), v_2), \\ \text{and} & L(v; t) = l_1(v_1; t) + l_2(v_2; t), \end{array} \tag{8}$$

where $H = \left(L^2(\Omega)\right)^2$ endowed with the scalar product $(u, v)_H = \left((u_1, v_1)^2_{0,\Omega} + (u_2, v_2)^2_{0,\Omega}\right)^{1/2}$, $V = H^1_{0,\Gamma_m}(\Omega) \times H^1_{0,\Gamma_u}(\Omega)$ and $\|v\|_V = (\|v_1\|^2_{1,\Omega} + \|v_2\|^2_{1,\Omega})^{1/2}$ is the norm associated with space $V$. It is easy to show that the linear form $L(\cdot)$ is continuous and that the bilinear form $a(\cdot, \cdot)$ is continuous and $V$-elliptic. Then, by using a theorem formulated by J.L. Lions (see [18]), the weak problem (8) admits a unique solution.

We now consider the following energy-like functional:

$$E(\eta, \tau) = \int_0^D \int_\Omega k(x) \left(\nabla u_1(\eta, t) - \nabla u_2(\tau, t)\right)^2\ dx\ dt + \frac{1}{2} \int_\Omega \left(u_1(\eta, D) - u_2(\tau, D)\right)^2\ dx, \tag{9}$$

and the following minimization problem:

$$\left\{ \begin{array}{l} (\eta^*, \tau^*) = \underset{(\eta,\tau)\in\mathscr{U}}{\operatorname{argmin}} E(\eta, \tau), \quad \mathscr{U} = L^2\left(0, D; H^{-1/2}_{00}(\Gamma_u) \times H^{1/2}_{00}(\Gamma_u)\right), \\ \text{with } u_1 \text{ and } u_2 \text{ solutions of (3) and (4) respectively.} \end{array} \right. \tag{10}$$

By using the convexity of space $\mathscr{U}$, the existence and uniqueness of the solution to the Cauchy problem in the case of compatible data allows proving that the solution $(\eta^*(t), \tau^*(t))$ of the minimization problem (10), if it exists and is unique, is the solution of the data completion problem ($i.e.(\eta^*, \tau^*) = (\varphi, \xi)$).

# 4 Discretization method and error estimation

Let $X_h$ be the finite element space for which the following classical assumptions are verified:

(i) $\Omega$ is a polyhedral domain in $\mathbb{R}^d, d = 2, 3$.

(ii) $\mathscr{T}_h$ is a regular triangulation of $\bar{\Omega}$ i.e. $h = \max_{K\in\mathscr{T}_h} h_K \to 0$ and $\max_{K\in\mathscr{T}_h} \dfrac{h_K}{\rho_K} \leq c$ being a constant independent of $h$, $h_K$ is the element, $K$ the diameter and $\rho_K$ the inscribed circle diameter of $K$.

(iii) $\Gamma_u$ and $\Gamma_m$ can be written exactly as the merger of the faces of several finite elements $K \in \mathscr{T}_h$.

(iv) The family $(K, P_K, \Sigma_K), K \in \mathscr{T}_h$ for all $h$ is affine-equivalent to a reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ of $\mathscr{C}^0$ regularity.

(v) The following inclusion is satisfied: $P_l(\hat{K}) \subset \hat{P} \subset H^1(\hat{K})$ for $l \geq 1$.

These assumptions imply that $X_h \subset H^1(\Omega)$. We define the following spaces:

$$\begin{aligned} X_{uh} &= \{v_h \in X_h; v_{h|\Gamma_u} = 0\}, \\ X_{mh} &= \{v_h \in X_h; v_{h|\Gamma_m} = 0\}, \end{aligned}$$

and $V_h = X_{mh} \times X_{uh} \subset V$ the finite dimensional approximation space.

Given $u_{0h} \in V_h$ the $V_h$-interpolation of $u_{00}$, the semi-discrete problem associated with (8) is written as

$$\begin{aligned}
&\text{Find } u_h(t) = \big(u_{1h}(t), u_{2h}(t)\big) \in L^2(0, D; V_h) \text{ such that} \\
&\quad \frac{d}{dt}\big(u_h(t), v_h\big)_H + a\big(u_h(t), v_h\big) = L(v_h; t), \quad \forall\, v_h = (v_{1h}, v_{2h}) \in V_h \ \text{ in } \mathscr{D}'(]0, D[) \qquad (11)\\
&\quad u_h(0) = u_{0h}.
\end{aligned}$$

We now turn to time discretization. We introduce time step $\Delta t$ and time $t_n = n\Delta t$, $0 \le n \le N$, and denote the approximation of $u(\cdot, t_n)$ by $u_h^n \in V_h$. This gives the discrete problem based on the backward Euler scheme:

$$\begin{aligned}
&\text{Find } \{u_h^n \in V_h;\ 0 \le n \le N\} \text{ such that} \\
&\quad \frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h)_H + a(u_h^{n+1}, v_h) = L(v_h; t_{n+1}), \quad 0 \le n \le N-1, \quad \forall\, v_h \in V_h \qquad (12)\\
&\quad u_h^0 = u_{0h}.
\end{aligned}$$

The same argument as for the weak problem (8) provides the existence and uniqueness of the solutions of (11) and (12).

Using the standard procedure described in [19], we report the following error estimate:

**Proposition 4.1** *In addition to the assumptions stated above, let us assume that the integer $l \ge 1$ exists such that the following inclusion is satisfied:*

$$H^{l+1}(\hat{K}) \subset \mathscr{C}^s(\hat{K}) \text{ with continuous injection} \qquad (13)$$

*where $s$ is the maximal order of the partial derivatives occurring in the definition of the set $\hat{\Sigma}$.*

*Then, if the solution $u$ of the variational problem (8) also verifies $u(t) \in \big(H^{l+1}(\Omega)\big)^2$ for all $t \in [0, D]$, there is a constant $C$ independent on $h$ and $\Delta t$ such that*

$$\begin{aligned}
\|u(t_n) - u_h^n\|_V \le C \Bigg\{ h^l \Bigg( |||u_{00}\,|||_{l+1,\Omega} + \left( \int_0^{t_n} |||\frac{du}{dt}(s)\,|||_{l+1,\Omega}^2 \ ds \right)^{1/2} \Bigg) + \\
+\Delta t \left( \int_0^{t_n} |||\frac{d^2u}{dt^2}(s)\,|||_{l+1,\Omega}^2 \ ds \right)^{1/2} \Bigg\}, \qquad (14)
\end{aligned}$$

*where $\{u_h^n \in V_h, 0 \le n \le N\}$ is the discrete solution.*

# 5 Noisy data, error estimates and stopping criterion

## 5.1 Error estimates and data noise effects

In the case of given perturbed data, say $(\phi^\delta, T^\delta)$, problem (12) is written as:

$$\begin{aligned}
&\text{Find } \{u_{h\delta}^n \in V_h;\ 0 \le n \le N\} \text{ such that} \\
&\quad \frac{1}{\Delta t}(u_{h\delta}^{n+1} - u_{h\delta}^n, v_h)_H + a(u_{h\delta}^{n+1}, v_h) = L^\delta(v_h; t_{n+1}), \quad 0 \le n \le N-1, \quad \forall\, v_h \in V_h \qquad (15)\\
&\quad u_{h\delta}^0 = u_{0h},
\end{aligned}$$

where $L^\delta(\cdot)$ is given by the expression of $L(\cdot)$ with $(\phi, T)$ being replaced by $(\phi^\delta, T^\delta)$.

**Proposition 5.1** *Under assumptions of proposition 4.1, if the solution $u$ of the variational problem (8) also verifies $u(t) \in \big(H^{l+1}(\Omega)\big)^2$ for all $t \in [0, D]$, then there is an independent constant $C$ on*

*$h$, $\Delta t$ and data such that*

$$\|u(t_n)-u^n_{h\delta}\|_V \le C\left\{h^l\left(|||u_{00}\,|||_{l+1,\Omega}+\left(\int_0^{t_n}|||\frac{du}{dt}(s)\,|||^2_{l+1,\Omega}\ \ ds\right)^{1/2}\right)+\right.$$
$$\left.+\Delta t\left(\int_0^{t_n}|||\frac{d^2u}{dt^2}(s)\,|||^2_{l+1,\Omega}\ \ ds\right)^{1/2}+\sqrt{\Delta t}\left(\sum_{j=1}^n\left\|\frac{d(T-T^\delta)}{dt}(t_j)\right\|_{1/2,00,\Gamma_m}+\frac{1}{h}\|\delta(t_j)\|\right)\right\},\quad (16)$$

*where*

$$\|\delta(t)\|=\left(\|T(t)-T^\delta(t)\|^2_{1/2,00,\Gamma_m}+\|\phi(t)-\phi^\delta(t)\|^2_{-1/2,00,\Gamma_m}\right)^{1/2} \quad (17)$$

*is the norm of the data noise and $\{u^n_{h\delta}\in V_h;\ 0\le n\le N\}$ is the solution of the discrete problem (15) associated with the noisy Cauchy problem.*

**Proof** We write $u(t_n)-u^n_{h\delta}=u(t_n)-u^n_h+u^n_h-u^n_{h\delta}=\rho^n+\theta^n_\delta$. An estimation for $\|\rho^n\|_V$ is given immediately by the proposition 4.1.

From (12) and (15), we obtain

$$\frac{1}{\Delta t}(\theta^{n+1}_\delta-\theta^n_\delta,v_h)_H+a(\theta^{n+1}_\delta,v_h)=L^\delta(v_h;t_{n+1})-L(v_h;t_{n+1}),\quad \forall\ v_h\in V_h. \quad (18)$$

$\omega^n_\delta\in V_h$ is defined by

$$(\omega^n_\delta,v_h)_H=L^\delta(v_h;t_{n+1})-L(v_h;t_{n+1})=\left(\frac{d(\bar u_1-\bar u^\delta_1)}{dt}(t_n),v_h\right)_H$$
$$+\,a\left(\bar u_1(t_n)-\bar u^\delta_1(t_n),v_h\right)+\langle\phi(t_n)-\phi^\delta(t_n),v_{2h}\rangle_{-1/2,00,\Gamma_m}, \quad (19)$$

where $\bar u^\delta_1(t)$ is the lifting of the extended Dirichlet conditions with noise $T^\delta(t)$. By using trace and lifting operator properties and inverse inequalities, we prove that

$$\|\omega^n_\delta\|_H\le C\left(\left\|\frac{d(T-T^\delta)}{dt}(t_n)\right\|_{1/2,00,\Gamma_m}+\frac{1}{h}\|\delta(t_n)\|\right). \quad (20)$$

Choosing $v_h=\dfrac{\theta^{n+1}_\delta-\theta^n_\delta}{\Delta t}$ in (18), it gives

$$\|\theta^n_\delta\|^2_V\le C\Delta t\sum_{j=1}^n\|\omega^j_\delta\|^2_H. \quad (21)$$

and then, with (20),

$$\|\theta^n_\delta\|^2_V\le C\Delta t\sum_{j=1}^n\left(\left\|\frac{d(T-T^\delta)}{dt}(t_n)\right\|_{1/2,00,\Gamma_m}+\frac{1}{h}\|\delta(t_n)\|\right)^2. \quad (22)$$

Therefore, by using (20) and (22) we obtain an estimation of $\|\theta^{n+1}_\delta\|_V$ that leads to (16). ■

## 5.2 Stopping criterion for the minimization process

When noise is introduced in the Cauchy data, we observe during the optimization process that the error reaches a minimum before increasing very fast, leading to a numerical explosion. At the same time, the energy-like functional asymptotically attains a minimal threshold, which is a strictly positive constant dependent on the noise. It is noteworthy that this constant vanishes in the case of compatible Cauchy data. The aim now is to theoretically determine this threshold in order to propose a stopping criterion dependent on the noise rate and which allows stopping the minimization process just before the numerical explosion.

We introduce a general quadrature formula where nodes and weights are denoted by $(t_j, \alpha_j)$ to approximate the integral of a continuous function $f$ on the time interval,

$$I_N(f) = \sum_{j=0}^{N} \alpha_j f(t_j) \sim \int_0^D f(t)\, dt. \tag{23}$$

The noisy discrete functional is then given by

$$E_h^\delta(\eta,\tau) = \sum_{j=0}^{N} \alpha_j \int_\Omega k(x) \left(\nabla u_{1h\delta}^j(\eta) - \nabla u_{2h\delta}^j(\tau)\right)^2 \, dx + \frac{1}{2}\int_\Omega \left(u_{1h\delta}^N(\eta) - u_{2h\delta}^N(\tau)\right)^2 \, dx. \tag{24}$$

**Proposition 5.2** *Under the assumptions of proposition 4.1, if the solution $u$ of the variational problem (8) also verifies $u(t) \in \left(H^{l+1}(\Omega)\right)^2$ for all $t \in [0, D]$ and if $(\eta^*, \tau^*)$ is the solution of the minimization problem (10), then there is an independent constant $C$ on $h$ and the data such that*

$$\begin{aligned} E_h^\delta(\eta^*,\tau^*) \le C \Bigg\{ & h^{2l} \sum_{n=0}^{N} \alpha_n \left( |\!|\!| u_0 |\!|\!|_{l+1,\Omega} + \left( \int_0^{t_n} |\!|\!| \frac{du}{dt}(s) |\!|\!|_{l+1,\Omega}^2 \, ds \right)^{1/2} \right)^2 + \\ & + \Delta t^2 \left( \sum_{n=0}^{N} \alpha_n \int_0^{t_n} |\!|\!| \frac{d^2u}{dt^2}(s) |\!|\!|_{l+1,\Omega}^2 \, ds \right) + \\ & \Delta t \sum_{n=0}^{N} \alpha_n \left( \sum_{j=1}^{n} \left\| \frac{d(T-T^\delta)}{dt}(t_j) \right\|_{1/2,00,\Gamma_m} + \frac{1}{h} \|\delta(t_j)\| \right)^2 \Bigg\}. \end{aligned} \tag{25}$$

**Proof** Let $(\eta^*, \tau^*)$ be the solution of the minimization problem (10) with compatible Cauchy data. After several algebraic operations and taking into account the fact that $u_1(\eta^*;t) = u_2(\tau^*;t), \ \ \forall t \in [0, D]$, we can write

$$\begin{aligned} E_h^\delta(\eta^*,\tau^*) = & \sum_{j=0}^{N} \alpha_j \int_\Omega k(x) \left[ \left( \nabla u_{1h\delta}^j(\eta^*) - \nabla u_1(\eta^*;t_j) \right) - \left( \nabla u_{2h\delta}^j(\tau^*) - \nabla u_2(\tau^*;t_j) \right) \right]^2 \, dx + \\ & + \frac{1}{2} \int_\Omega \left[ \left( u_{1h\delta}^N(\eta^*) - u_1(\eta^*, t_N) \right) - \left( u_{2h\delta}^N(\tau^*) - u_2(\tau^*;t_N) \right) \right]^2 \, dx. \end{aligned} \tag{26}$$

It follows that

$$\begin{aligned} E_h^\delta(\eta^*,\tau^*) \le 2\|k\|_{L^\infty(\Omega)} & \sum_{j=0}^{N} \alpha_j \left( |u_{1h\delta}^j(\eta^*) - u_1(\eta^*;t_j)|_{1,\Omega}^2 + |u_{2h\delta}^j(\tau^*) - u_2(\tau^*;t_j)|_{1,\Omega}^2 \right) + \\ & + \left( \|(u_{1h\delta}^N(\eta^*) - u_1(\eta^*;t_N)\|_{0,\Omega}^2 + \|u_{2h\delta}^N(\tau^*) - u_2(\tau^*;t_N)\|_{0,\Omega}^2 \right), \end{aligned} \tag{27}$$

and then, ther exists a costant $C$ that may depend on $\Delta t$, such that

$$E_h^\delta(\eta^*,\tau^*) \le C \sum_{j=0}^{N} \alpha_j \|u_{h\delta}^j(\eta^*,\tau^*) - u(\eta^*,\tau^*;t_j)\|_V^2. \tag{28}$$

Therefore, using proposition 5.1, we derive (25). ■

Nevertheless, at each time step except the last, the error semi-norm in $V$ is majorized by the error norm in $V$. Then, this overestimation must be taken into acocunt to be more precise. We prove that

$$\|u_{h\delta}^n(\eta^*,\tau^*) - u(t_n;\eta^*,\tau^*)\|_H^2 \ge \frac{1}{2}\|u_{1h\delta}^n(\eta^*) - u_{2h\delta}^n(\tau^*)\|_{0,\Omega}^2 \tag{29}$$

Hence, when the noisy discrete functional (24) reaches its minimum and, when $h$ and $\Delta t$ are sufficiently small, we obtain through (25) and (29),

$$E_h^\delta(\eta^*, \tau^*) \sim O\left(S_\delta(\eta^*, \tau^*)\right)), \tag{30}$$

where

$$S_\delta(\eta^*, \tau^*) = \Delta t \sum_{n=0}^{N} \alpha_n \left( \left( \sum_{j=1}^{n} \left\| \frac{d(T - T^\delta)}{dt}(t_j) \right\|_{1/2,00,\Gamma_m} + \frac{1}{h} \|\delta(t_j)\| \right)^2 - \frac{1}{2} \|u_{1h\delta}^n(\eta^*) - u_{2h\delta}^n(\tau^*)\|_{0,\Omega}^2 \right). \tag{31}$$

In order to propose a stopping criterion based on these theoretical estimates, let us use $(\eta^j, \tau^j)$ to denote the unknowns and $E_j$ the value of the discrete noisy functional at the $j$-th iteration.

At first, the stopping criterion relies on verifying that the noisy discrete functional has reached $S_\delta$. Taking into account the asymptotical behavior of the functional, we want to stop the optmization algorithm when the functional variations become lower than the functional itself, and then lower than $S_\delta$. Moreover, the ratio $E_j/E_{j-1} < 1$ tends to 1. Then, multiplying $S_\delta$ by this ratio, the threshold $S_\delta$ is weakened before the asymptote is reached by the functional. Thus a consistent stopping criterion based on the description of the behavior of $E_h^\delta(\cdot, \cdot)$ and the estimation (25), could be

$$\max\{E_j, |E_j - E_{j-1}|\} \leq \frac{E_j}{E_{j-1}} S_\delta(\eta_j, \tau_j). \tag{32}$$

# 6 Numerical issues

## 6.1 Numerical procedure

Let us describe the calculation method of the elements required for the optimization procedure, more specifically the gradient of the functional. We assume that the triangulation $\mathscr{T}_h$ of $\Omega$ is characterized by $n$ nodes. Let $p$ and $q$ denote the number of nodes on the boundaries $\Gamma_u$ and $\Gamma_m$ respectively and $(\omega_i)_{1\leq i\leq n} = (\omega_{1i}, \omega_{2i})_{1\leq i\leq n}$ the which is the canonical basis of $V_h$. We write the unknowns $\eta(t_n)$ and $\tau(t_n)$ as $X_\eta^n$ and $X_\tau^n$ respectively. Vectors $U_1^n$ and $U_2^n$ correspond to fields $u_1(t_n)$ and $u_2(t_n)$ respectively, vectors $\mathrm{T}^n$ and $\Phi^n$ correspond to the Dirichlet and Neumann data $T(t_n)$ and $\phi(t_n)$ respectively. We introduce the following notations, $(K_1)_{kl} = a_1(\omega_{1k}, \omega_{1l})$, $(K_2)_{kl} = a_2(\omega_{2k}, \omega_{2l})$, $(F_1^n)_k = l_1(\omega_{1k}; t_n)$ and $(F_2^n)_k = l_2(\omega_{2k}; t_n)$ depending on the Neumann data $\phi(t_n)$. As the bi-linear forms are similar, we note $K = K_1 = K_2$.

The linear systems associated with (3) and (4) respectively are given by:

$$\begin{cases} \left(\dfrac{M}{\Delta t} + K\right) U_1^{n+1} = F_1^{n+1}(X_\eta^{n+1}) + \dfrac{M}{\Delta t} U_1^n \\ L_m U_1^{n+1} = \mathrm{T}^{n+1} \\ U_1^0 = U_0, \end{cases} \tag{33}$$

$$\begin{cases} \left(\dfrac{M}{\Delta t} + K\right) U_2^{n+1} = F_2^{n+1}(\Phi^{n+1}) + \dfrac{M}{\Delta t} U_2^n \\ L_u U_2^{n+1} = X_\tau^{n+1} \\ U_2^0 = U_0. \end{cases} \tag{34}$$

The functional can be written as follows:

$$E(X_\eta, X_\tau) = \frac{1}{2} \sum_{n=0}^{N} \alpha_n (U_1^n - U_2^n)^t K (U_1^n - U_2^n) + (U_1^N - U_2^N)^t M (U_1^N - U_2^N) \tag{35}$$

We want to calculate the functional derivatives with respect to each component with index $i$ and at each time step $k$ of the two unknowns $X_\eta$ and $X_\tau$ written as $X_\eta^{i,k}$ and $X_\tau^{i,k}$ respectively.

First, we derive the functional with respect to the unknown $X_\eta$.

$$\frac{\partial E}{\partial X_\eta^{i,k}}(X_\eta, X_\tau) = 2\sum_{n=k}^{N} \alpha_n \left(\frac{\partial U_1^n}{\partial X_\eta^{i,k}}\right)^t K(U_1^n - U_2^n)$$
$$+ \delta_k^N \left(\frac{\partial U_1^N}{\partial X_\eta^{i,N}}\right)^t M(U_1^N - U_2^N), \quad k = 1, ..., N; \quad i = 1, ..., p. \quad (36)$$

In the sequel, we denote by $U_{1,i,k}^n$ the derivative of $U_1^n$ with respect to $X_\eta^{i,k}$. By deriving the linear system (33), $U_{1,i,k}^{n+1}$ is the solution of:

$$\left(\frac{M}{\Delta t} + K\right) U_{1,i,k}^{n+1} = \frac{\partial F_1^{n+1}}{\partial X_\eta^{i,k}} + \frac{M}{\Delta t} U_{1,i,k}^n, \quad (37)$$

As $\mathrm{T}^n$ and $U_0$ are independent on $X_\eta^{n,k}$, their derivatives vanish. Moreover,

$$U_{1,i,k}^n = 0 \text{ if } k > n \text{ and } \frac{\partial F_1^n}{\partial X_\eta^{i,k}} = 0 \text{ if } k \neq n.$$

We note $\tilde{F}_{1,i}^n = \dfrac{\partial F_1^n}{\partial X_\eta^{i,n}} = (\delta_i^j)_{1 \le j \le m}$. We have then

$$\begin{cases} \left(\dfrac{M}{\Delta t} + K\right) U_{1,i,k}^{n+1} = \delta_{n+1}^k \tilde{F}_{1,i}^{n+1} + \mathbb{1}_{\{k>n+1\}} \dfrac{M}{\Delta t} U_{1,i,k}^n \\ L_m U_{1,i,k}^{n+1} = 0 \\ U_{1,i,k}^0 = 0, \quad 1 \le i \le m, \quad 0 \le n \le N-1, \quad 1 \le k \le N. \end{cases} \quad (38)$$

Now, we derive the functional with respect to the unknown $X_\tau$.

$$\frac{\partial E}{\partial X_\tau^{i,k}}(X_\eta, X_\tau) = -\sum_{n=k}^{N} \alpha_n \left(\frac{\partial U_2^n}{\partial X_\tau^{i,k}}\right)^t K(U_1^n - U_2^n)$$
$$- \delta_k^N \left(\frac{\partial U_2^N}{\partial X_\tau^{i,N}}\right)^t M(U_1^N - U_2^N), \quad k = 1, ..., N; \quad i = 1, ..., p. \quad (39)$$

In the sequel, we denote by $U_{2,i,k}^n$ the derivative of $U_2^n$ with respect to $X_\tau^{i,k}$. By deriving the linear system (34), $U_{2,i,k}^{n+1}$ is the solution of

$$\left(\frac{M}{\Delta t} + K\right) U_{2,i,k}^{n+1} = \frac{\partial F_2^{n+1}}{\partial X_\tau^{i,k}} + \frac{M}{\Delta t} U_{2,i,k}^n, \quad (40)$$

As $F_2^n(\Phi^n)$ and $U_0$ are independent on $X_\tau^{n,k}$, their derivatives vanish. Moreover,

$$U_{2,i,k}^n = 0 \text{ if } k > n \text{ and } \frac{\partial X_\tau^n}{\partial X_\tau^{i,k}} = 0 \text{ if } k \neq n.$$

We note $\tilde{X}_{\tau,i}^n = \dfrac{\partial X_\tau^n}{\partial X_\tau^{i,n}} = (\delta_i^j)_{1 \le j \le m}$. We then have

$$\begin{cases} \left(\dfrac{M}{\Delta t} + K\right) U_{2,i,k}^{n+1} = \mathbb{1}_{\{k>n+1\}} \dfrac{M}{\Delta t} U_{2,i,k}^n \\ L_u U_{2,i,k}^{n+1} = \tilde{X}_{\tau,i}^{n+1} \\ U_{2,i,k}^0 = 0, \quad 1 \le i \le m, \quad 0 \le n \le N-1, \quad 1 \le k \le N. \end{cases} \quad (41)$$

Here, we consider the case of real applications where we have only measured and noisy data $(T^\delta, \phi^\delta)$ given with a noise rate $0 < a < 1$. We are therefore not able to calculate exactly the norm of the difference between the exact and noisy data involved in the stopping criterion (32). We must therefore estimate these norms. This is done as follows:

$$T(x,t) - aT(x,t) \leq T^\delta(x,t) \leq T(x,t) + aT(x,t), \quad \forall\, x \in \Gamma_m \tag{42}$$

$$\Longleftrightarrow \quad \frac{-a}{1-a}T^\delta(x,t) \leq T(x,t) - T^\delta(x,t) \leq \frac{a}{1+a}T^\delta(x,t) \tag{43}$$

$$\text{and then} \quad \|T(t) - T^\delta(t)\|^2_{1/2,00,\Gamma_m} \leq \max\left\{\frac{a}{1-a}, \frac{a}{1+a}\right\} \|T^\delta(t)\|_{1/2,00,\Gamma_m} \tag{44}$$

Proceeding in the same way for the Neumann data and the time derivative of the Dirichlet data, we have:

$$\|\delta(t)\| \leq \frac{a}{1-a}\left(\|T^\delta(t)\|^2_{1/2,00,\Gamma_m} + \|\phi^\delta(t)\|^2_{-1/2,00,\Gamma_m}\right)^{1/2} \tag{45}$$

and

$$\left\|\frac{d(T-T^\delta)}{dt}(t)\right\|_{1/2,00,\Gamma_m} \leq \frac{a}{1-a}\left\|\frac{dT^\delta}{dt}(t)\right\|_{1/2,00,\Gamma_m}. \tag{46}$$

The stopping criterion (32) can then be written as follows:

$$\max\{E_j, |E_j - E_{j-1}|\} \leq \frac{E_j a^2 \Delta t}{E_{j-1}(1-a)^2}\sum_{n=1}^{N}\alpha_n\left(\sum_{k=1}^{n}\|\frac{dT^\delta}{dt}(t_k)\|_{1/2,00,\Gamma_m}\right.$$
$$\left. + \frac{1}{h}\left(\|T^\delta(t_k)\|^2_{1/2,00,\Gamma_m} + \|\phi^\delta(t_k)\|^2_{-1/2,00,\Gamma_m}\right)\right)^2 - \frac{1}{2}\sum_{n=1}^{N}\alpha_n(U_1^n - U_2^n)^t M(U_1^n - U_2^n). \tag{47}$$

## 6.2 Numerical results

We consider the following Cauchy problem on the domain $\Omega$ given by figure (2):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = 0 \text{ in } \Omega\times]0,1[ \\ u = g \text{ on } \Gamma_m\times]0,1[ \\ \nabla u\cdot n = h \text{ on } \Gamma_m\times]0,1[ \\ u(\cdot,0) = u_0 \text{ in } \Omega, \end{cases} \tag{48}$$

where $g$, $h$ and $u_0$ are the Cauchy data extracted from the exact solution that we intend to approximate.

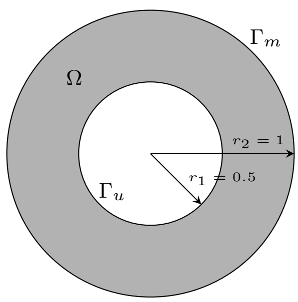



Figure 2: Ring

### 6.2.1 Axisymmetric example

The Cauchy problem is written in polar coordinates $(\rho, \theta)$ and we assume that its solution does not depend on the angular coordinate. The state equation of (48) is therefore

$$\frac{\partial u(\rho,t)}{\partial t} - \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)u(\rho,t) = 0. \tag{49}$$

An analytical solution of this equation is given by $u(\rho, t) = e^{-t} J_0(\rho)$ where $J_0(\cdot)$ is the Bessel function of the first kind of order 0.

Figure 3 represents the finite element discretization error with respect to the maximum edge size of the mesh. We choose a sufficiently small $\Delta t$ such that time discretization is negligible. Similarly, figure 4 shows the time discretization error with respect to the time step, by considering $h$ sufficiently small. These results are in agreement with the theoretical error estimate (14).

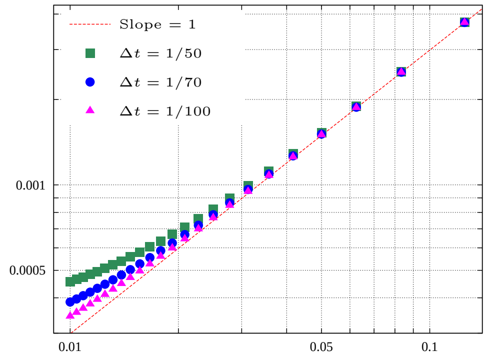



Figure 3: Evolution of $\|u(t_N) - u_h^N\|_V$ with respect to $h$ for different $\Delta t$ and for the axisymmetric example.

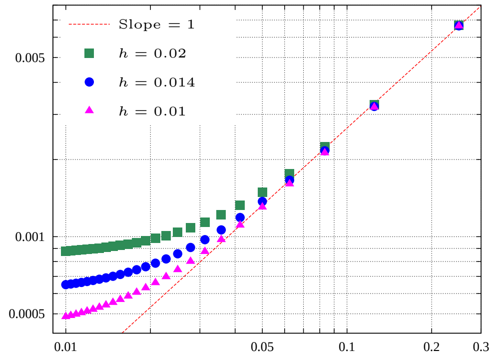



Figure 4: Evolution of $\|u(t_N) - u_h^N\|_V$ with respect to $\Delta t$ for different $h$ and for the axisymmetric example.

#### 6.2.2 Two dimensional example

We now consider the resolution of the Cauchy problem (48) in two dimensions. An analytical solution of this problem is given by $u(x, y; t) = e^{-2t} cos(x + y)$ which provides Cauchy data on $\Gamma_m$.

The figure 5 represents the discrete solution of the Cauchy problem along with the selected points $p_1$, $p_2$ and $p_3$ used to represent the time evolution of the solution to the data completion problem. Figures 6 and 7 represent the solution and the discrete solution of the data completion problem obtained by using energy-like method. We can see that the temperature and heat flux recovered are close to the exact ones.

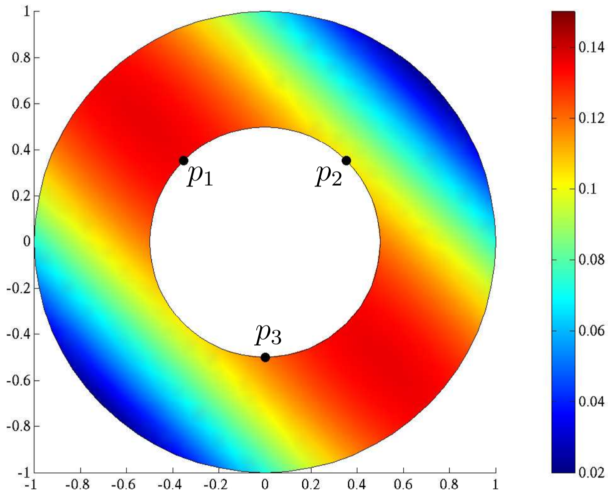
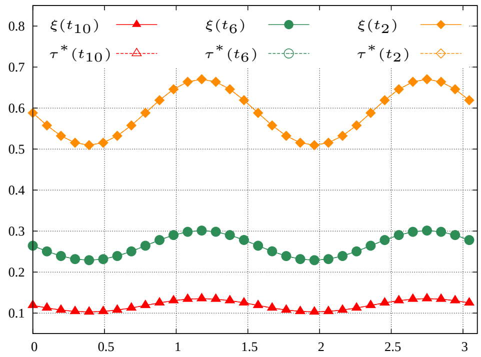
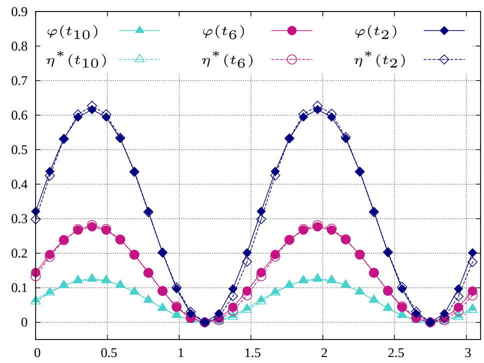



Figure 5: Exact temperature and selected points $p_1$, $p_2$ and $p_3$, $h = 0.1$, $\Delta t = 0.1$.

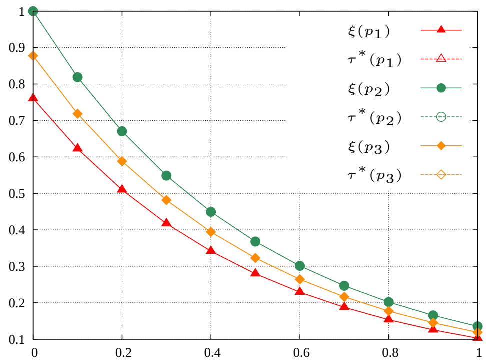



(a) Temperature

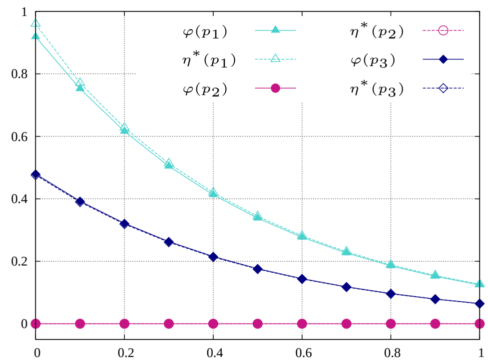



(b) Heat flux

Figure 6: Exact (filled markers) and identified (empty markers) temperature and heat flux on $\Gamma_u$ at times $t_2 = 0.2$, $t_6 = 0.6$ and $t_{10} = 1$ for the 2D example, $h = 0.1$, $\Delta t = 0.1$.




(a) Temperature




(b) Heat flux

Figure 7: Time evolution of exact (filled markers) and identified (empty markers) temperature and heat flux on selected points $p_1$, $p_2$ and $p_3$ on $\Gamma_u$ for the 2D example, $h = 0.1$, $\Delta t = 0.1$.

Figure 9 represents the finite element discretization error with respect to the maximum edge size of the mesh with a sufficiently small $\Delta t$. Similarly, figure 8 shows the time discretization error with respect to the time step with $h$ sufficiently small.

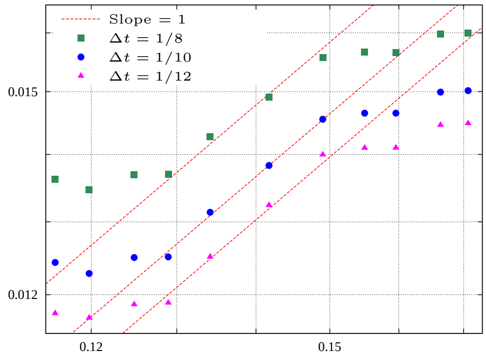



Figure 8: Evolution of $\|u(t_N) - u_h^N\|_V$ with respect to $\Delta t$ for different $h$ for the 2D example.

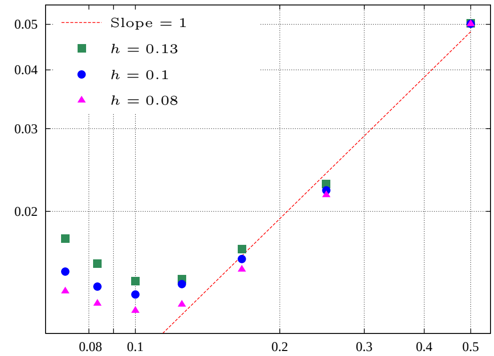



Figure 9: Evolution of $\|u(t_N) - u_h^N\|_V$ with respect to $h$ for different $\Delta t$ for the 2D example.

These results are in agreement with the theoretical error estimate (14). Nevertheless, the optimization process is significantly perturbed when the discretization steps $h$ and $\Delta t$ are not of the same order. Indeed, since the error associated with the largest discretization step behaves like numerical noise, the energy-like method does not provide a discrete solution with the required accuracy. Moreover, since the results obtained in the axisymmetric case unambiguously confirm the theoretical estimate, this numerical noise could also be related to a mesh effect.

We introduce a Gaussian random noise on data with an amplitude depending on a rate $a$. Figures 10 and 11 represent the error and the energy-like functional at each iteration of the optimization process for different noise rates. These behaviors make it necessary to introduce a criterion to stop the optimization process before numerical explosion.

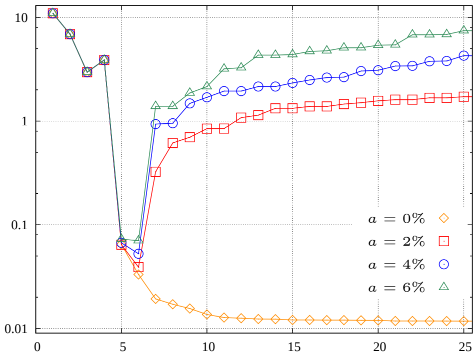



Figure 10: Evolution of $\|u(t_N) - u_{h\delta}^N\|_V$ during the optimization process for different noise rates and for the 2D example.

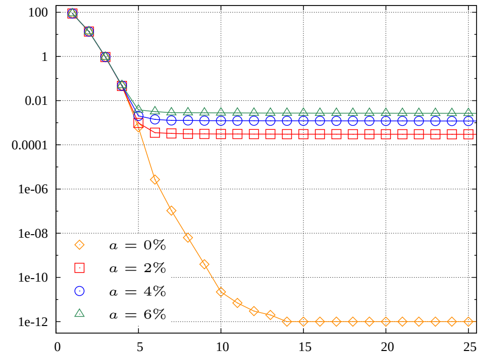



Figure 11: Evolution of $E_h^\delta(\eta, \tau)$ during the optimization process for different noise rates and for the 2D example.

Next, we choose $h$ and $\Delta t$ such that the discretization error is negligible in comparison to the error due to noise and we observe error and functional behaviors with respect to the satisfactory noise measurements. These noise measurements correspond to terms dependent on the data in the estimates (16) and (25). They are denoted by

$$m_\delta^n = \sqrt{\Delta t} \left( \sum_{j=1}^{n} \left\| \frac{d(T - T^\delta)}{dt}(t_j) \right\| + \frac{1}{h} \|\delta(t_j)\| \right) \quad \text{and} \quad M_\delta = \sum_{n=0}^{N} \alpha_n \left(m_\delta^n\right)^2. \tag{50}$$

These results, shown in figure 13, are in agreement with the error estimates (16) and (25).

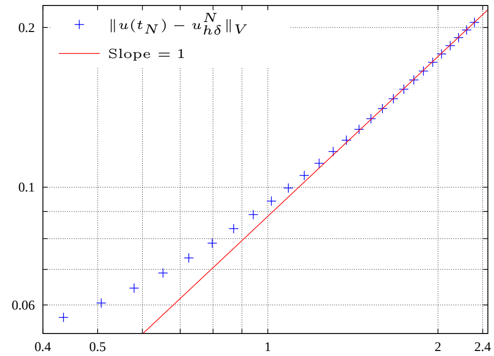



Figure 12: Evolution of $\|u(t_N) - u^N_{h\delta}\|_V$ with respect to $m^N_\delta$ for different noise rates and for the 2D example, $h = 0.09$, $\Delta t = 1/12$.

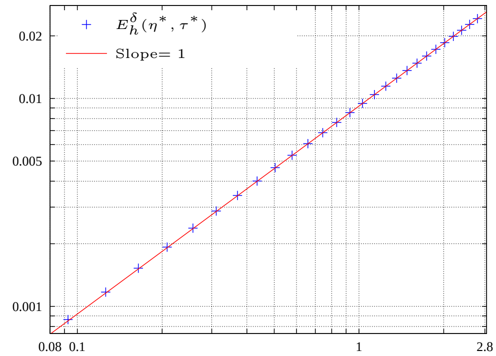
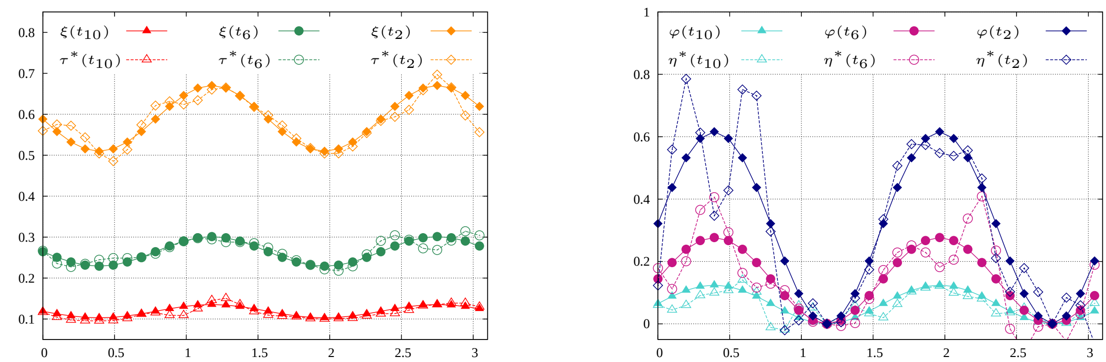



Figure 13: Evolution of $E^\delta_h(\eta^*, \tau^*)$ with respect to $M_\delta$ for different noise rates and for the 2D example, $h = 0.09$, $\Delta t = 1/12$.

As illustrated by figures 14, 15 and 16, the proposed stopping criterion allows identifying a consistent solution.




(a) Temperature

(b) Heat flux

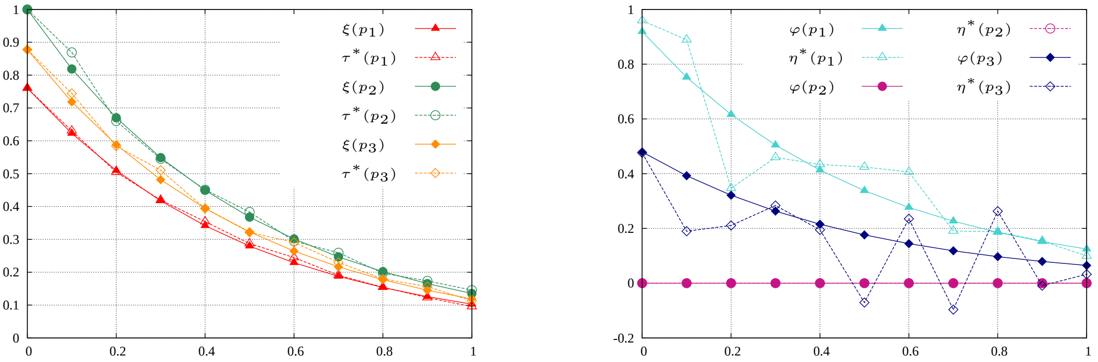
Figure 14: Exact (filled markers) and identified (empty markers) temperature and heat flux on $\Gamma_u$ at times $t_2 = 0.2$, $t_6 = 0.6$ and $t_{10} = 1$ for the 2D example, $h = 0.1$, $\Delta t = 0.1$, $a = 5\%$.

### 6.2.3 Stratified inner fluid problem

We now explore the efficiency of the proposed stopping criterion on the stratified inner fluid problem already studied in [9, 14]. We therefore consider the reconstruction of temperature and flux in a pipeline of infinite length. This application is used in several industrial processes. Indeed, knowledge of the temperature on the internal wall of a pipeline is necessary for controlling material safety: stratified inner fluid generates mechanical stresses which may cause damage such as cracks. We assume that the temperature does not depend on the longitudinal coordinate and then consider the following problem on the geometry defined by figure 17 :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = 0 \text{ in } \Omega \\ k \nabla u \cdot n + \alpha u = 20 \text{ on } \Gamma_m \\ k \nabla u(x,t) \cdot n + \alpha u(x,t) = 250.\, \mathbb{1}_{\{\Gamma_{u,up(t)}\}}(x) + 50.\, \mathbb{1}_{\{\Gamma_{u,lo(t)}\}}(x) \text{ on } \Gamma_u \\ u(.,0) = u_0 \end{cases} \tag{51}$$

where $k = \tilde{k}/\rho c$, $\alpha = \tilde{\alpha}/\rho c$ with $k = 17\ W.m^{-1}.K^{-1}$ is the constant thermal conductivity, $\tilde{\alpha} = 12$




(a) Temperature (b) Heat flux

Figure 15: Time evolution of exact (filled markers) and identified (empty markers) temperature and heat flux at selected points $p_1$, $p_2$ and $p_3$ on $\Gamma_u$ for the 2D example, $h = 0.1$, $\Delta t = 0.1$, $a = 5\%$.

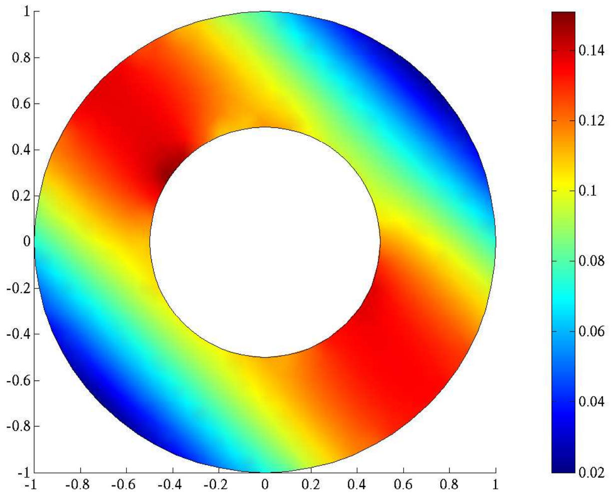


Figure 16: Identified solution for the 2D example, $a = 5\%$, $h = 0.1$, $\Delta t = 0.1$.

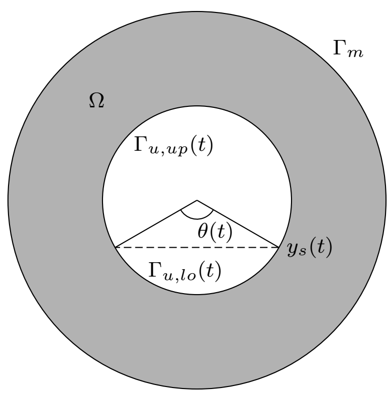



Figure 17: Stratified ring

on $\Gamma_m$ and 1000 on $\Gamma_u$ is the Fourier coefficient, $\rho$ and $c$ are the density and the heat capacity such that $\rho c = 1$. The radius of the inner and outer circles in figure 17 are the same as in figure 2. The boundary $\Gamma_u$ is partitioned into two parts, the lower arc $\Gamma_{u,lo}(t) = \{(x, y) \in \Gamma_u; y < y_s(t)\}$ and the upper arc $\Gamma_{u,up}(t) = \{(x, y) \in \Gamma_u; y \geq y_s(t)\}$. Angle $\theta(t)$ evolves linearly from 0 to $\pi$ with time. Therefore, the upper and lower parts of $\Gamma_u$, and then temperature on $\Gamma_u$, depend on $t$. The initial condition $u_0$ is the stationary solution of the problem (51) with $\theta = 0$.

The Cauchy data are generated by solving the forward problem defined by (51). Then, a random noise with a rate of $a = 5\%$ is introduced in the Dirichlet data while we assume that the flux is known exactly on $\Gamma_m$. The results presented here are obtained by using the proposed stopping criterion. Figure 18 represents the temperature field that has to be identified along with the selected points $p_1$, $p_2$ and $p_3$ used to represent the time evolution of the solution of the data completion problem. On the one hand, figure 19 shows the temperature and heat flux recovered in comparison to the solution of the data completion problem given by the numerical resolution of (51) at times $t_2 = 0.2, t_6 = 0.6$ and $t_{10} = 1$. On the other hand, figure 20 shows the time evolution of the temperature and heat flux recovered in comparison to the time evolution of the data completion problem solution on points $p_1$, $p_2$, $p_3$. Figure 21 represents the temperature field identified relating to figure 18. It should be noted that the reconstructed field is close to the field to be recovered. Finally, figure 22 represents the solution of the generic optimization algorithm. A numerical explosion without the proposed stopping criteria can be clearly observed in this case.

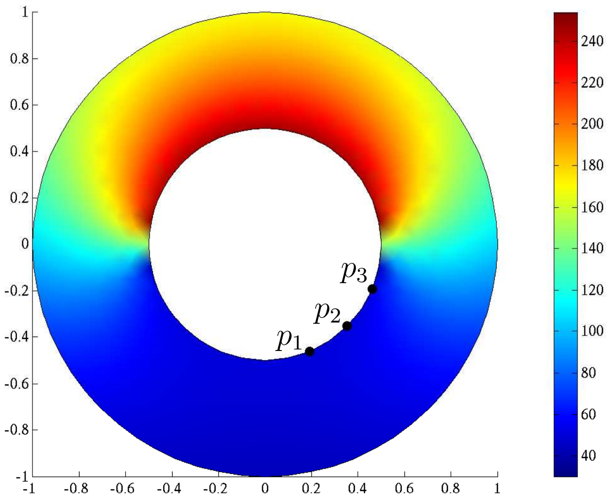
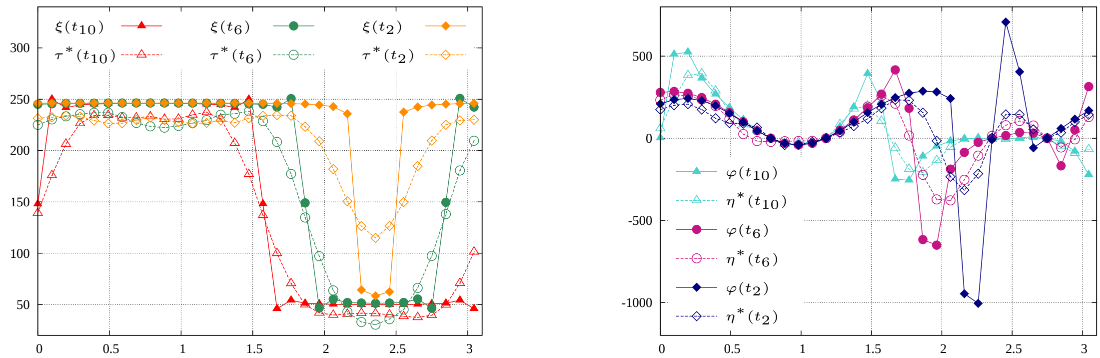



Figure 18: Exact temperature and selected points $p_1$, $p_2$ and $p_3$ for the stratified inner fluid example at time $t = 1$, $h = 0.1$, $\Delta t = 0.1$.

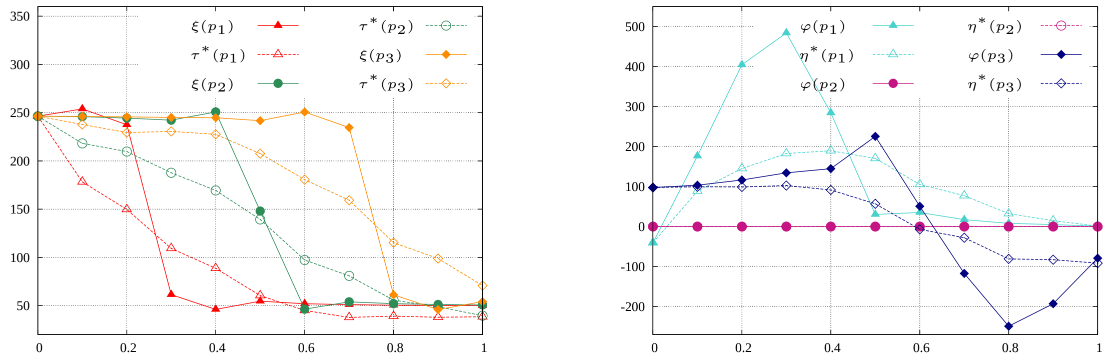
# 7 Conclusion

In this work, we stated the Cauchy problem as being the minimization of an energy-like functional and presented classical theoretical results. Then, we gave the finite element discretization and performed numerical convergence analysis. A priori error estimates were derived by taking into account the effects of noisy data. A stopping criterion was then proposed, dependent on the noise rate in order to control the numerical instability of the minimization process due to noisy data. A numerical procedure was proposed and numerical experiments performed to confirm the theoretical error estimates. Finally, we illustrated the robustness and efficiency of the proposed stopping criterion, especially in the case of singular data. It would now be interesting to couple this approach with a regularization procedure.




(a) Temperature (b) Heat flux

Figure 19: Exact (filled markers) and identified (empty markers) temperature and heat flux on $\Gamma_u$ at times $t_2$, $t_6$ and $t_{10}$ for the stratified inner fluid example, $h = 0.1$, $\Delta t = 0.1$, $a = 5\%$.




(a) Temperature (b) Heat flux

Figure 20: Time evolution of exact (filled markers) and identified (empty markers) temperature and heat flux on selected points $p_1$, $p_2$ and $p_3$ on $\Gamma_u$ (cf. figure 18) for the stratified inner fluid example, $h = 0.1$, $\Delta t = 0.1$, $a = 5\%$.

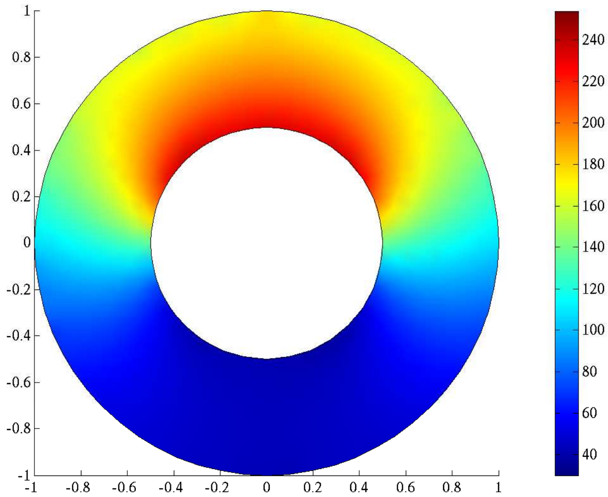


Figure 21: Identified solution for the stratified inner fluid example using the proposed stopping criterion at time $t = 1$, $h = 0.1$, $\Delta t = 0.1$, $a = 5\%$.

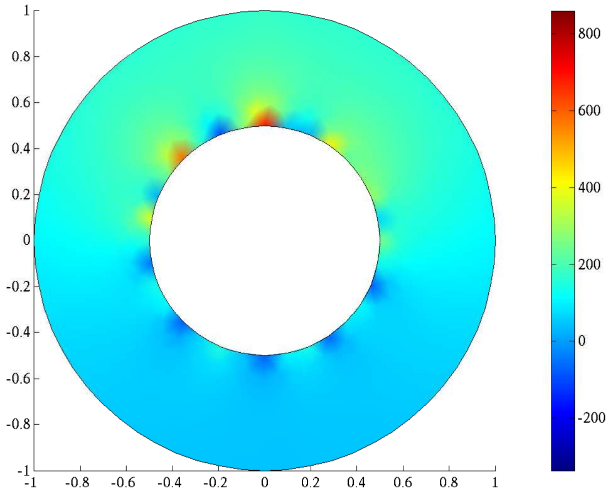


Figure 22: Identified solution for the stratified inner fluid example without using the proposed stopping criterion at time $t = 1$, $h = 0.1$, $\Delta t = 0.1$, $a = 5\%$.

## References


[1] Hadamard, J., 1953. *Lectures on Cauchy's problem in Linear Partial Differential Equation.* New York : Dover.

[2] Baumeister, J., and Leitão, A. "On iterative methods for solving ill-posed problems modeled by partial differential equations".

[3] Tikhonov, A., and Arsénine, V., 1976. *Méthode de résolution de problèmes mal posés.* Editions Mir.

[4] Cimetière, A., Delvare, F., Jaoua, M., and Pons, F., 2001. "Solution of the Cauchy problem using iterated tikhonov regularization". *Inverse Problems,* **17**(3), pp. 553–570.

[5] Bourgeois, L., 2005. "A mixed formulation of quasi-reversibility to solve the Cauchy problem for laplace's equation". *Inverse Problems,* **21**(3), pp. 1087–1104.

[6] Marin, L., 2005. "Numerical solution of the Cauchy problem for steady-state heat transfer in two-dimensional functionally graded materials". *International Journal of Solids and Structures,* **42**(15), pp. 4338–4351.

[7] Marin, L., 2009. "The minimal error method for the Cauchy next term problem in linear elasticity. Numerical implementation for two-dimensional homogeneous isotropic linear elasticity". *International Journal of Solids and Structures,* **46**(5), pp. 957–974.

[8] Chakib, A., and Nachaoui, A., 2006. "Convergence analysis for finite element approximation to an inverse Cauchy problem". *Inverse Problems,* **22**(4), pp. 1191–1206.

[9] Andrieux, S., Baranger, T. N., and Ben Abda, A., 2006. "Solving Cauchy problems by minimizing an energy-like functional". *Inverse Problems,* **22**(1), pp. 115–133.

[10] Baranger, T. N., and Andrieux, S., 2007. "An optimization approach for the Cauchy problem in linear elasticity". *Structural and Multidisciplinary Optimization,* **35**(2), pp. 141–152.

[11] Andrieux, S., and Baranger, T. N., 2008. "An energy error-based method for the resolution of the Cauchy problem in 3d linear elasticity". *Computer Methods in Applied Mechanics and Engineering,* **197**, pp. 902–920.

[12] Andrieux, S., and Baranger, T. N., 2008. "Energy methods for Cauchy problems of evolutions equations". *6th International Conference on Inverse Problems in Engineering : Theory and Practice, Journal of physics : Conference Series 135.*

[13] Baranger, T., and N.Andrieux, S. "Constitutive law gap functionals to solve Cauchy problem for a linear elliptic pde: a review". *http://hal.archives-ouvertes.fr/hal-00489572/fr/.*

[14] Rischette, R., Baranger, T. N., and Debit, N., 2011. "Numerical analysis of an energy-like minimization method to solve Cauchy problem with noisy data". *Journal of Computanional and Applied Mathematics,* **235**, pp. 3257–3269.

[15] Isakov, V., 1997. *Inverse problems for partial differential equations.* Springer.

[16] Lions, J. L., and Dautray, R., 1999. *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 2 - Functional and Variational Methods. Springer.

[17] Fursikov, A. V., 2000. *Optimal control of distributed systems.* American mathematical Society.

[18] Lions, J. L., 1968. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* Dunod.

[19] Thomée, V., 1980. *Galerkin Finite Element Methods for Parabolic Problems.* Springer-Verlag.

[20] Lions, J. L., and Magenes, E., 1968. *Problèmes aux limites non homogènes et applications*, Vol. 1. Dunod.